5.5a local stability of first order systems

Monday, March 22, 2021 11:03 AM

Recall: We can a Taylor's Thm to approximate a nonlinear system.

(viz Vid 2.8)

Thm: Let $\dot{X}(t) = F(X(t))$ be a system of autonomous 1st-order ODEs
$X(t) = (x_1(t), \ldots, x_n(t))^T$, $F = (f_1, \ldots, f_n)^T$, $f_i = f_i(x_1, \ldots, x_n)$.

Let $\bar{X}$ be an equilibrium of the system. Then the linearization of the system about $\bar{X}$ and letting $U(t) = X(t) - \bar{X}$ gives a system

$$\dot{U}(t) = J\,U(t),$$

where J is the Jacobian matrix of F at $\bar{X}$,

$$J(\bar{X}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Assume all partial derivatives are continuous in an open neighborhood of $\bar{X}$

Then $\bar{X}$ is locally asymp stable if $Re(\lambda_i) < 0$ $\forall$ eigenvalues λ_i and unstable if some $Re(\lambda_i) > 0$.

proof. sketch

$$\dot{X}(t) = F(X) \approx \underbrace{F(\bar{X})}_{0} + \underbrace{J(\bar{X})}_{\text{Jacobian}}(X(t)-\bar{X}) + \cancel{\tfrac{1}{2}(X(t)-\bar{X})^T \underbrace{H(\bar{X})}_{\text{Hessian}} (X(t)-\bar{X}) + \cdots}$$

$\Rightarrow$ $\dot{X}(t) \approx J(\bar{X})(X(t)-\bar{X})$ for $X(t)$ sufficiently close to $\bar{X}$

$$\dot{U}(t) = J(\bar{X})\,U(t)$$

$$U(t) = e^{tJ(\bar{X})}\,U(0).$$

Let $PBP^{-1} = J(\bar{X})$, where B is in Jordan canonical form, $B = \Lambda + N$, where Λ is a diagonal matrix, and N only has nonzero entries directly above the diagonal (some of which are 1, corresponding to Jordan blocks, and some of which are 0).

Then $e^{tJ(\bar{x})} = P\exp(t\Lambda + tN)P^{-1}$
$= P\exp(t\Lambda)\exp(tN)P^{-1}$

Note that N is nilpotent, and so is tN.
Thus, the power series of $\exp(tN)$ can be cut after n terms, so
$$\exp(tN) = \sum_{m=0}^{n-1} t^m C_m, \quad \text{where } C_m \in \mathbb{C}^{n\times n} \text{ has no dependence on } t.$$
$$" \frac{1}{m!}N^m."$$

Also, $\exp(t\Lambda)$ is a diagonal matrix with terms $e^{\lambda_i t}$.
But $\lim_{t\to\infty} e^{-rt}\cdot t^m = 0$ for any $r>0$ and integer $m<n$.

Thus, $\lim_{t\to\infty} \exp(t\Lambda)\exp(tN) = 0$.
$\Rightarrow \lim_{t\to\infty} U(t) = \lim_{t\to\infty} \exp(tJ(\bar{x}))U_0 = 0.$
$\Rightarrow \lim_{t\to\infty} X(t) = \bar{X}$, so locally asymptotically stable.

If any $Re(\lambda_i) > 0$, then $\lim_{t\to\infty} U(t) = \infty$, if $U(0) = V_i$, the associated eigenvector.
Of course, the linearization may break down as $|X(t) - \bar{X}|$ grows, but $X(t)$ will still leave a sufficiently small ball around $\bar{X}$, so it is unstable. $\square$

Def. 5.6 Let $\dot{x} = F(x)$ have equilibrium $\bar{x}$, and let J be the Jacobian of F at $\bar{x}$. Then $\bar{x}$ is **hyperbolic** if all eigenvalues of J have nonzero real part and **nonhyperbolic** otherwise.

Thm 5.4 Let $\dot{x} = F(x)$, $x \in \mathbb{R}^2$, and the partial derivatives of F are continuous in an open neighborhood of an equilibrium $\bar{x}$. Then $\bar{x}$ is locally asymptotically stable if

$$Tr(J) < 0 \text{ and } \det(J) > 0,$$

where J is the Jacobian at $\tilde{X}$. It is unstable if either $Tr(J) > 0$ or $\det(J) < 0$.